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# ON THE MOTION OF CHAPLYGIN'S SPHERE ON A HORIZONTAL PLANE*

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The motion of a heavy sphere on a fixed horizontal plane is considered. It is assumed that the centre of mass of the sphere is at its geometric centre, while the principal central moments are different (Chaplygin's sphere). Using the method of averaging, the motion of the sphere is investigated under slip conditions when there is low viscous and also low dry friction. It is shown that when the sphere moves with viscous friction it tends, for the majority of initial data, to rotate about the longest of the axes of the principal central moments of inertia. The motion of the sphere centre tends to become uniform so that the slip velocity approaches zero exponentially. A system of averaged equations, which is fully integrable, is obtained in the case of almost equal moments of inertia, when the friction is dry. The solutions are analyzed.

1. Suppose that, due to the action of an initial push, a heavy sphere moves on a stationary horizontal plane, touching it at a single point of its surface. The geometric centre of the sphere coincides with the centre of mass, and the principal central moments of inertia are generally different ($A > B > C$) /1/. We select the stationary system of coordinates $O_1X_1X_2X_3$ so that $O_1X_1X_2$ is in the supporting plane on which the sphere moves, and the $X_3$ axis is directed vertically upward.

Let $x_1, x_2, x_3$ be the coordinates of the centre of the sphere $O$ in the stationary system of coordinates. Then $x_3 \equiv R$, where $R$ is the radius of the sphere, and $x_1$ and $x_2$ satisfy the differential equation

$$Mx_1^{\cdot\cdot} = F_1, \quad Mx_2^{\cdot\cdot} = F_2 \tag{1.1}$$

where $M$ is the mass of the sphere, and $F_1$ and $F_2$ are the components of the force of friction along the $X_1$ and $X_2$ axes.

For the viscous friction we have

$$F_1 = -fMV_1, \quad F_2 = -fMV_2 \tag{1.2}$$

where $f$ is the coefficient of friction, and $V_1$ and $V_2$ are the projections on the $X_1$ and $X_2$ axes of the absolute velocity $\mathbf{V}$ of the point of the sphere which touches the supporting plane.

Let $G_1, G_2, G_3$ be the projections on the $X_1, X_2, X_3$ axes, respectively, of the sphere angular momentum vector $\mathbf{G}$ relative to its centre. All forces applied to the sphere intersect the $X_3$ axis, hence $G_3 = G_{30} = \text{const}$. The equations of the angular momenta in projections on the $X_1$ and $X_2$ axes are

$$G_1^{\cdot} = F_2R, \quad G_2^{\cdot} = -F_1R \tag{1.3}$$

Taking into account (1.2) we can rewrite (1.1) and (1.3) as follows:

$$x_1^{\cdot\cdot} = -fV_1, \quad x_2^{\cdot\cdot} = -fV_2, \quad G_1^{\cdot} = -fMRV_2, \quad G_2^{\cdot} = fMRV_1 \tag{1.4}$$

We will denote by $\omega_1$ and $\omega_2$ the projections of the sphere absolute angular velocity vector $\omega$ on the $X_1$ and $X_2$ axes, respectively. Then

$$V_1 = x_1^{\cdot} - \omega_2R, \quad V_2 = x_2^{\cdot} + \omega_1R \tag{1.5}$$

The expression for the derivative of kinetic energy $T$ of the sphere motion relative to its centre of mass can be written in the form

$$T^{\cdot} = fMR\,(\omega_2V_1 - \omega_1V_2) \tag{1.6}$$

Equations (1.4) have two first integrals

$$G_1 - MRx_2^{\cdot} = K_1 = \text{const}, \quad G_2 + MRx_1^{\cdot} = K_2 = \text{const} \tag{1.7}$$

which are corollaries of the conservation of angular momentum relative to the contact point.

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Substituting the expressions for $x_1^{\cdot}$ and $x_2^{\cdot}$ from (1.7) into (1.6), and into the last two equations of (1.4), we reduce (1.4) and (1.6) to the form

$$G_i^{\cdot} = -f(G_i - K_i + \omega_i MR^2);\ i = 1, 2 \tag{1.8}$$
$$\varkappa^{\cdot} = 2f\{(G_1 - K_1)(\varkappa G_1 - \omega_1) + (G_2 - K_2)(\varkappa G_2 - \omega_2) + MR^2[\varkappa(\omega_1 G_1 + \omega_2 G_2) - \omega_1^2 - \omega_2^2]\}/G^2,\ \varkappa = 2T/G^2$$

We denote by $a$, $b$, $c$ quantities inverse to $A, B, C$ $(a < b < c)$. Obviously $a \leqslant \varkappa \leqslant c$. This inequality defines in the space of variables $G_1, G_2, \varkappa$ a region representing the space contained between the two planes which correspond to the sphere rotation about the axis of maximum $(\varkappa = a)$ and minimum $(\varkappa = c)$ moment of inertia.

Let the friction be low, i.e. $f$ is a small parameter. Then in the unperturbed problem $(f = 0)$ the sphere centre moves rectilinearly and uniformly, the principal angular momentum vector $\mathbf{G}$ is constant, and $\varkappa = \text{const}$, and the sphere performs Euler-Poinsot motion about its centre. According to the Poinsot geometrical representation of the motion of a solid /2/ the quantity $\varkappa$ is equal to the square of the distance from the sphere centre the plane tangent to the ellipsoid of inertia and normal to $\mathbf{G}$. Rotations of the sphere about the major, minor, and mean axes of the ellipsoid of inertia correspond to $\varkappa = c$, $\varkappa = a$, $\varkappa = b$.

We will investigate the perturbed motion using the method of averaging. Averaging the right-hand sides of (1.8) with respect to the unperturbed Euler-Poinsot motion, we obtain (retaining the previous notation for the slow variables)

$$dG_i/d\tau = -(G_i - K_i + \langle\omega_i\rangle MR^2);\ i = 1, 2$$
$$d\varkappa/d\tau = 2\{(G_1 - K_1)(\varkappa G_1 - \langle\omega_1\rangle) + (G_2 - K_2)\times (\varkappa G_2 - \langle\omega_2\rangle) + MR^2[\varkappa(\langle\omega_1\rangle G_1 + \langle\omega_2\rangle G_2) - \langle\omega_1^2\rangle - \langle\omega_2^2\rangle]\}/G^2$$

where $\tau = ft$ and the angle brackets denote averaging over rapid variables, as functions of time and the slow variables $G_1, G_2, \varkappa$.

The averaging procedure was described in detail in /3/, where averaging over the Euler-Poinsot motion in the non-resonant case was carried out for the first time. Calculations showed that

$$\langle\omega_1\rangle = \varkappa G_1,\ \langle\omega_2\rangle = \varkappa G_2,\ \langle\omega_1^2 + \omega_2^2\rangle = \varkappa^2(G^2 - G_{30}^2) - {}^1/_2(G^2 + G_{30}^2)\lambda(\varkappa)$$
$$\lambda(\varkappa) = (\varkappa - a)(\varkappa - c) - h[1 - E(k)/K(k)]k^{-2}$$
$$h = \begin{cases} (b-c)(\varkappa - a), & \varkappa < b; \\ (b-a)(\varkappa - c), & \varkappa > b; \end{cases} \qquad k^2 = \begin{cases} \dfrac{c-b}{b-a}\,\dfrac{\varkappa - a}{c - \varkappa}, & \varkappa < b \\ \dfrac{b-a}{c-b}\,\dfrac{c-\varkappa}{\varkappa - a}, & \varkappa > b \end{cases}$$

where $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kind, respectively.

We thus obtain the averaged system of equations

$$dG_i/d\tau = -G_i(1 + \varkappa MR^2) + K_i;\ i = 1, 2 \tag{1.9}$$
$$\frac{d\varkappa}{d\tau} = MR^2\left(1 + \frac{G_{30}^2}{G^2}\right)\lambda(\varkappa)$$

Let us consider the last of Eqs.(1.9). The use of the modulus of the elliptic functions $k$ as the slow variable when investigating the equations of the perturbed motion, averaged over the Euler-Poinsot motion, was proposed in /4, 5/. On the basis of an analysis of the averaged equation for $k^2$, conclusions were drawn on the evolution of the motion.

In the present paper we use the variable $\varkappa$, which is related to $k^2$ by a simple formula. Although the equations for $k^2$ in /4, 5/ and in this paper (if we change from $\varkappa$ to $k^2$) are different, the method of analysis is the same. It is based on the use of the properties of complete elliptic integrals of the first and second kind. We shall briefly enumerate the properties of the functions $\lambda(\varkappa)$ that are required for a qualitative investigation of (1.9). The function $\lambda(\varkappa)$ is defined in $[a, c]$, if it is additionally defined at the points $a$, $b$, $c$ with respect to continuity. Moreover $\lambda(\varkappa) < 0$ everywhere, except at the points $a$, $b$, $c$, where it vanishes. On the basis of asymptotic expansions of $\lambda(\varkappa)$ at the points $a$, $b$, $c$ and its properties enumerated above (taking into account the fact that $1 \leqslant 1 + G_{30}^2/G^2 \leqslant 2$ ), it is possible to draw the following conclusions: if at the initial instant of time $\varkappa_0 \in (a, b)$, then in the course of time $\varkappa$ decreases and approaches $a$: if, however, $\varkappa_0 \in (b, c)$, then $\varkappa$ will decrease and reach the value of $b$ in a finite time. Thus, the solutions of the averaged system reach the separatrix and pass through it. However, near the separatrix the method of averaging cannot be used in its usual form.

The problem of passing through the separatrix is dealt with in /6, 7/. It is shown there that the motion of a dynamic system is defined by averaged equations until the separatrix is reached. As regards the further behaviour we can speak with definite probability. The measure of the set of initial data, for which the motion after crossing the separatrix cannot be defined using the averaged system together with the perturbations, is small.

The problem considered here of the evolution of the motion of the sphere is described for most of initial data with an accuracy of $o(f|\ln f|)$ by the solutions of an averaged system "joined together" from solutions in different regions, i.e. from solutions in the regions $a < \varkappa < b$ and $b < \varkappa < c$. All solutions of the averaged system obtained by joining approach equilibrium in the plane $\varkappa = a$ as $\tau \to +\infty$, i.e. the limit motion of the sphere is rotation about the axis of the maximum of the principal central moments of inertia.

Since $\varkappa \to a$ as $\tau \to +\infty$, from the first two of equations of (1.9) it can be seen that

$$G_i - K_i/\Lambda \to 0,\ \tau \to +\infty,\ i = 1, 2;\ \Lambda = 1 + MR^2/A \tag{1.10}$$

and it can be verified that the characteristic numbers of the functions $G_i(\tau) - K_i/\Lambda$ are equal to $\Lambda$.

From (1.7) we find that as $t \to +\infty$

$$x_1^{\cdot} \to K_2R/A\Lambda,\ x_2^{\cdot} \to -K_1R/A\Lambda \tag{1.11}$$

Since the final motion of the sphere is rotation about the axis of maximum moment of inertia $A$, we have $\boldsymbol{\omega} \to \mathbf{G}/A$ as $t \to +\infty$, i.e.

$$\omega_i \to K_i/A\Lambda;\ i = 1,\ 2;\ \omega_3 \to G_{30}/A \tag{1.12}$$

It follows from (1.10)–(1.12) that the slip velocity approaches zero exponentially.

Thus the final motion of the sphere for any initial data from region $a < \varkappa < b$ and for the majority of initial data from region $b < \varkappa < c$ is such that its centre moves uniformly along a straight line, and the sphere itself rotates at constant angular velocity about the axis of maximum moment of inertia, while the slip velocity approaches zero exponentially.

2. As before, let the sphere centre of mass be at its geometric centre, and let the motion occur with slip when there is low dry friction (i.e. the coefficient of friction $f$ is a small quantity). The moments of inertia $A$, $B$, $C$ are different but close to each other. As in Sect.1, we use the notation $O_1X_1X_2X_3$ for the stationary system of coordinates; the axes of the attached system of coordinates $Oz_1z_2z_3$ are directed along the principal central axes of inertia; $x_1, x_2$ are the coordinates of the centre of the sphere in the stationary coordinate system; $p$, $q$, $r$ are the projections of the sphere instantaneous angular velocity $\boldsymbol{\omega}$ on the $z_1, z_2, z_3$ axes; and $a_{ij}$ $(i, j = 1, 2, 3)$ are the direction cosines that determine the transition from the attached to the fixed system of coordinates. The quantities $(A - B)/I_0$, $(B - C)/I_0$, $(A - C)/I_0$, $f$ are of the same order of smallness. Here $I_0 = 2MR^2/5$ is the moment of inertia of the homogeneous sphere. The coordinates of the point of contact in the attached system are $-Ra_{31}$, $-Ra_{32}$, $-Ra_{33}$.

The theorems on the variation of the momentum and the angular momentum, and the Poisson kinematic relations enable us to write the following system of equations in $x_1, x_2, p, q, r, a_{ij}$

$$\begin{gathered}
x_1^{\cdot\cdot} = -fg\cos\alpha,\ x_2^{\cdot\cdot} = -fg\sin\alpha \\
Ap^{\cdot} + (C - B)\,qr = fMgR\,(-a_{11}\sin\alpha + a_{21}\cos\alpha) \\
Bq^{\cdot} + (A - C)\,pr = fMgR\,(-a_{12}\sin\alpha + a_{22}\cos\alpha) \\
Cr^{\cdot} + (B - A)\,pq = fMgR\,(-a_{13}\sin\alpha + a_{23}\cos\alpha) \\
a_{i1}^{\cdot} = a_{i2}r - a_{i3}q,\ a_{i2}^{\cdot} = a_{i3}p - a_{i1}r,\ a_{i3}^{\cdot} = a_{i1}p - a_{i2}q \\
(i = 1,\ 2,\ 3)
\end{gathered} \tag{2.1}$$

where $\alpha$ is the angle between the vector $\mathbf{V}$ (see Sect.1) and the $X_1$ axis.

We complement (2.1) by the following relations

$$V_1 = x_1^{\cdot} - R\,(pa_{21} + qa_{22} + ra_{23}),\ V_2 = x_2^{\cdot} + R\,(pa_{11} + qa_{12} + ra_{13}) \tag{2.2}$$

Equations (2.1) and (2.2) yield the first-approximation differential equations for $\alpha$ and $\mathbf{V}$ /8/

$$\begin{gathered}
\alpha^{\cdot} = (\Phi_1\cos\alpha + \Phi_2\sin\alpha)\,/\,V,\ V^{\cdot} = -7fg/2 + (\Phi_1\sin\alpha - \Phi_2\cos\alpha) \\
\Phi_i = \frac{B - C}{A}\,qra_{i1} + \frac{C - A}{B}\,pra_{i2} + \frac{A - B}{C}\,pqa_{i3};\quad i = 1, 2
\end{gathered} \tag{2.3}$$

As in /8, 9/ we introduce instead of $a_{i1}$, $a_{i2}$, $a_{i3}$ the variables $\rho_i$, $\zeta_i$, $\gamma_i$ using the formulas

$$a_{i1} = \rho_i\frac{q}{\sqrt{p^2+q^2}}\sin\gamma_i + \rho_i\frac{p}{\sqrt{p^2+q^2}}\frac{r}{\omega}\cos\gamma_i + \zeta_i\frac{p}{\omega}$$

$$a_{i2} = -\rho_i\frac{p}{\sqrt{p^2+q^2}}\sin\gamma_i + \rho_i\frac{q}{\sqrt{p^2+q^2}}\frac{r}{\omega}\cos\gamma_i + \zeta_i\frac{q}{\omega}$$

$$a_{i3} = -\rho_i \frac{\sqrt{p^2+q^2}}{\omega} \cos\gamma_i + \zeta_i \frac{r}{\omega}$$

where the quantities $\zeta_1, \zeta_2, \zeta_3$ are the cosines of the angles between the vector $\boldsymbol{\omega}$ and the $X_1, X_2, X_3$ axes, and the quantity $|R\zeta_3|$ is the distance of the centre of the sphere from the plane perpendicular to $\boldsymbol{\omega}$ and passing through the point of contact. The trivial integrals of the Poisson kinematic equations in the new variables have the form $\rho_i^2 + \zeta_i^2 = 1$.

In the unperturbed motion $\gamma_i^{\cdot} = \omega$, and the quantities $\zeta_i$ and $\rho_i$ are constant. We will make one more replacement of variables. Instead of $\zeta_1$ and $\zeta_2$ we will introduce $\alpha_1$ and $\alpha_2$ using the formulas

$$\alpha_1 = \zeta_1 \cos\alpha + \zeta_2 \sin\alpha, \quad \alpha_2 = -\zeta_1 \sin\alpha + \zeta_2 \cos\alpha$$

The quantities $\alpha_1$ and $\alpha_2$ have the following geometric meaning: $\alpha_1$ is the cosine of the angle between the vectors $\boldsymbol{\omega}$ and $\mathbf{V}$, and $\alpha_2$ is the cosine of the angle between $\boldsymbol{\omega}$ and the vector parpendicular to $\mathbf{V}$ lying in the horizontal plane, with the shortest notation from $\boldsymbol{\omega}$ to that vector being anticlockwise.

Equations (2.1) and (2.3) in the new variables are not given here for brevity. Note that the variables $x_1^{\cdot}, x_2^{\cdot}, p, q, r, \alpha_1, \alpha_2, \zeta_3, \alpha, V$ are slow and $\gamma_i$ are fast.

Averaging the right-hand sides of the equations for the slow variables with respect to the fast variables $\gamma_i$, we obtain the following first-approximation system:

$$x_1^{\cdot\cdot} = -fg\cos\alpha, \quad x_2^{\cdot\cdot} = -fg\sin\alpha \tag{2.4}$$

$$Ap^{\cdot} + (C-B)\,qr = fMgR\alpha_2 p/\omega \ \{ABC, \ pqr\} \tag{2.5}$$

$$\alpha_1^{\cdot} = -\frac{5}{2} fg \frac{\alpha_1\alpha_2}{\omega R}, \quad \alpha_2^{\cdot} = \frac{5}{2} fg \frac{1-\alpha_2^2}{\omega R} \tag{2.6}$$

$$\zeta_3^{\cdot} = -\frac{5}{2} fg \frac{\zeta_3\alpha_2}{\omega R}, \quad \alpha^{\cdot} = 0, \quad V^{\cdot} = -7fg/2 \tag{2.7}$$

From (2.5) we obtain the following equation for $\omega$:

$$\omega^{\cdot} = \frac{5}{2} fg \frac{\alpha_2}{R} \tag{2.8}$$

From (2.7) we obtain that the velocity of the point of contact to a first approximation, as in the case of a homogeneous sphere /2/, has a constant direction ($\alpha = \mathrm{const}$) in the stationary coordinate system, and its modulus decreases linearly and vanishes after a time $2V_0/7fg$. From that instant the sphere begins to roll, and hence it is necessary to solve the averaged system in that interval. In a time interval of the order of $1/f$ the solutions of the averaged system approximate to the solution of the exact system with an error $f$.

It follows from (2.4) that the centre of mass trajectory is, to a first approximation, a parabola.

Equations (2.6)–(2.8) have the following first integrals:

$$\alpha_1\omega = c_1, \quad \sqrt{1-\alpha_2^2}\,\omega = c_2, \quad \zeta_3\omega = c_3$$

and the general solution has the form

$$\alpha_2 = \frac{\chi t + c_4}{\varkappa}, \quad \alpha_1 = \frac{c_1}{c_2\varkappa}, \quad \zeta_3 = \frac{c_3}{c_2\varkappa}, \quad \omega = c_2\varkappa; \tag{2.9}$$

$$\chi = \frac{5fg}{2Rc_2}, \quad \varkappa = \sqrt{1+(\chi t + c_4)^2}$$

where the constants $c_1, \ldots, c_4$ are determined by the initial conditions.

Formulas (2.9) show that in the course of time $|\alpha_1| \to 0, |\alpha_2| \to 1, |\omega| \to \infty, |\zeta_3| \to 0$, i.e. the vectors $\boldsymbol{\omega}$ and $\mathbf{V}$ are so oriented, as to be orthogonal and lie in a horizontal plane.

Note that the quantities $p/\omega, q/\omega$ and $r/\omega$ can be calculated using the same formulas as for $p$, $q$, $r$ in the Euler-Poinsot motion where the role of time is played by the quantity /8/

$$\tau = \int_0^t \omega\, dt$$

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# BIFURCATION OF COMMON LEVELS OF FIRST INTEGRALS OF THE KOVALEVSKAYA PROBLEM*

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The structure of integral manifolds in the Kovalevskaya problem of a heavy solid about a fixed point is considered. An analytic definition of a bifurcation set is obtained, and bifurcation diagrams are constructed. The number of two-dimensional toruses that appear in the composition of the integral manifold is indicated for each connected component, additional to the bifurcation set in the space of first integral constants.

The solution of the problem of the motion of a solid about a fixed point, as formulated by Kovalevskaya /1/, has been dealt with in many publications. We shall mention only a few of them. Appel'rot was the first to identify four classes of motion of the Kovalevskaya gyroscope /2/. A more detailed study of particular motions appeared in /3/, where a geometric treatment of Appel'rot classes is presented as corresponding to parts of the surface of multiple roots of the Kovalevskaya polynomial in the space of first-integral constants. The hodograph was used in /4, 5/ for a complete study of the motion belonging to the first and second classes, and the so-called particularly unusual motion of the third class in which the moving hodograph of the angular velocity of the body is a closed curve.

The set of zero measure corresponds to Appel'rot classes in the space of first integral constants. The remaining classes were not studied to any great extent, and it is only recently that their important qualitative properties were established /6/. It was assumed that the first Euler-Poisson equations are independent of the motions considered. However, it is still not known exactly at what values of the constant integrals the latter are independent. It is proved below that the Appel'rot classes correspond to the cases of integral dependence. The study of this question enables us to indicate in all cases the number of connected components of the integral manifold, each of which in the space of Euler-Poisson variables is a two-dimensional torus that carries conditionally periodic motions /7, 8/. The fact that integral manifolds, that do not degenerate when the Poincare parameter approaches zero, consist of two toruses is pointed out in /6/.

The investigation of integral manifolds as part of the solution of the problem of the topological analysis of classical dynamic systems can be traced back to Poincare and Birkhoff. It was formulated in modern terms by Smail /9/.

Finally, we note /10/, where, with some inaccuracies, eliminated in /11/ when investigating general cases, the particular problem of the bifurcation of the integrals of energy and areas is solved. The Kovalevskaya integral, and hence the complete integrability of the system, were ignored.

1. Let $p, q, r$ be the components of the angular velocity vector $\boldsymbol{\omega}$, and $\nu_1, \nu_2, \nu_3$ the components of the unit vector $\boldsymbol{\nu}$ of the vertical in the trihedron accompanying the solid. By a suitable selection of the moving axes and units of measurement, we reduce the Euler-Poisson equations in the Kovalevskaya problem to the form

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